Lecture 17

Matroid polytope!

1) finish algo proof (see lec16)
2) TU proof
3) Facets

Next time: Matroid intersect.

T.U. Proof

Recall matroid polytope

$$\boxed{P_M = \operatorname{conv}(\{1_S : \underbrace{S \in \mathcal{I}\}}_{X})}.$$

Want to show $P_M = P$ where

$$\boxed{\begin{aligned} P = \{ x \in \mathbb{R}^E : & \\ \text{(rank}_S) \qquad & x(S) \le r(S) \ \forall S \subseteq E \\ \text{(nonnegativity)} \qquad & x_e \geq 0. \quad \}. \end{aligned}}$$

$$A \quad \begin{bmatrix} \\ - \ 1_S \ - \\ \\ \end{bmatrix}$$

T.U. proof

- Note that $X = \{$ integral points in $P \}$.

why? • $z \in \mathbb{Z}^E \cap P \Rightarrow z_i \in \{0,1\}$ $\quad$ (rank$_{S=\{i\}}$ + nonneg.)

• $1_S \in P \Rightarrow \underset{1_S(S)}{\underset{\shortparallel}{|S|}} \le r(S)$ $\quad$ (rank$_S$).

$r(S) \le |S|$ $\qquad$ $|S| = r(S) \Rightarrow S \in I$.

$\Rightarrow$ enough to show P integral i.e. LP = IP or equiv, all vertices of $P \in \mathbb{Z}^E$.

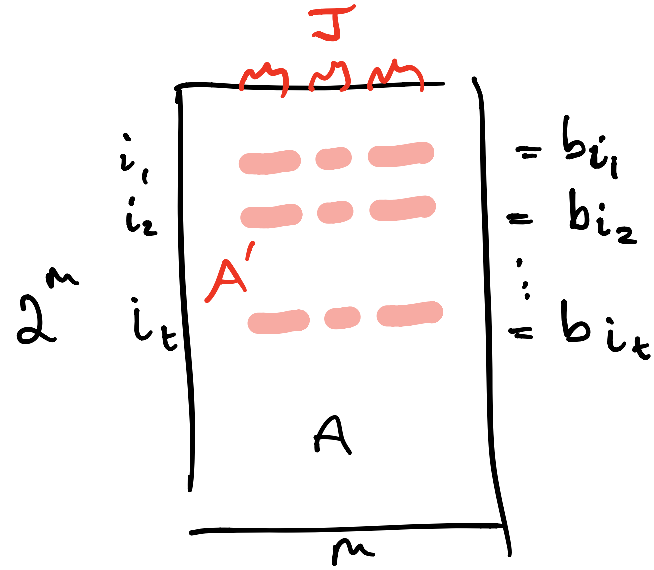
- If $P = \{x : Ax \le b, x \ge 0\}$, <u>is A TU?</u> <u>No</u>

$m = |E|$

of $P \subseteq \mathbb{R}^m$

- Recall that vertices come from m <u>tight</u> constraints.

(some from rows of A, some from $x \ge 0$).



▷ get vertices from setting $x_{E \setminus J} = 0$ solving $A' x_J = b'$ for remaining entries x_j.

- Instead of showing A T.U., show we can "<u>make</u>" A' T.U.

  $\Rightarrow x_J = (A')^{-1} b'$, b' integral.

  $x_{E \setminus J} = 0$

  $\Rightarrow$ by A' T.U., $x_J \in \mathbb{Z}^J$.

- In fact, submatrix A' will be even more special:

  rows of A' $\longleftrightarrow$ subsets of E,

  we can make the subsets form a chain $S_1 \subseteq \cdots \subseteq S_k$.

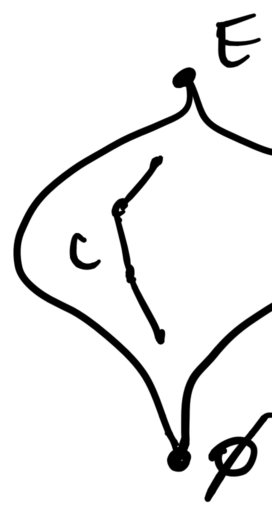
$$\Rightarrow \quad A' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & & 1 \end{bmatrix}$$

which is T.U. (exercise).

(or can see $A'x = b$ has integral solutions for b integral directly.).

- will also be helpful for matroid intersection!

- Actually, stronger:



Claim Let F be a face of P.
then $\exists$ chain $\mathcal{C}$, and subset $J \subseteq E$

$$F = \left\{ x \in \mathbb{R}^E : \begin{array}{ll} x(S) \le r(S) & \forall S \subseteq E \\ x(S) = r(S) & \forall S \in \mathcal{C} \\ x_e \ge 0 & \forall e \in J \\ x_e = 0 & \forall e \notin J, \end{array} \right\}.$$

(tight chain)

(set $x_{E \setminus J} \to 0$)

- use lemma from submodularity of rank.

Lemma: $\forall x \in P$, the tight constraints

$$\mathcal{T} := \{ S : x(S) = r(S) \} \subseteq 2^E$$

are closed under $\cap$ and $\cup$.

i.e. $R, S \in T$ i.e. $x(S) = r(S)$
$x(R) = r(R)$

then $S \cup R \in T$ i.e. $x(S \cup R) = r(S \cup R)$
$S \cap R \in T$ $x(S \cap R) = r(S \cap R)$.

Proof of claim from lemma:

- From polyhedra, we know

$$F = \{ x \in \mathbb{R}^E : \begin{array}{ll} x(S) \leq r(S) & \forall S \subseteq E \\ x(S) = r(S) & \forall S \in T \\ x_e \geq 0 & \forall e \in J, \\ x_e = 0 & \forall e \in E \setminus J \end{array} \}.$$

i.e. face comes from making some constraints tight.

- Enough to show can replace

T by chain C.

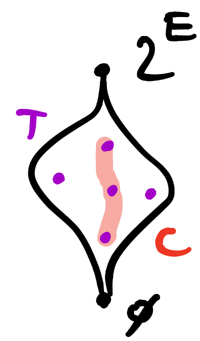
- "can replace" means they yield equivalent equalities.

$$\mathrm{span}(T) := \mathrm{span}(1_S : S \in T) = \mathrm{span}(1_S : S \in C) =: \mathrm{span}(C)$$

- To show, let C be a maximal <u>subchain</u> of T.

  i.e. $C \subseteq T$, C chain

  $\forall S \in T$, $\exists R \in C$ s.t. $S \not\subseteq R$ or $S \not\supseteq R$.



- We claim $\mathrm{span}(C) = \mathrm{span}(T)$

- Suppose

$\Rightarrow$

( ).

$\Rightarrow$ The set

$V(S) = \{ \quad \}$

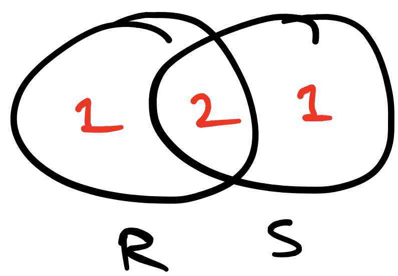
- Among all such S, take one with

( ).

- Let

- **Lemma** $\Rightarrow$

$\Rightarrow$

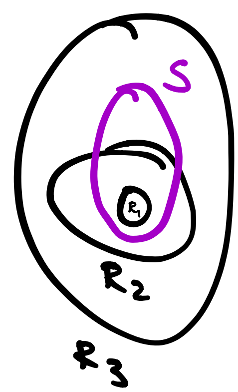


- Since

else

- Let

- But

  because

  (Exercise).

  and

  ( ).

$\Rightarrow$ $\square$

Corollary: Let x vertex of P.

$\Rightarrow$ (double check yourself).

Proof of lemma want to show

$T := \{ \quad \}$

closed under $\cap$ and $\cup$.

•

• $=$ (1)

$=$ (2)

$\leq$ (3)

$\leq$ (4)

- (1) because

- (2) AKA
  holds because

- (3) because

- (4) is

- $\bigstar$

$\Rightarrow$

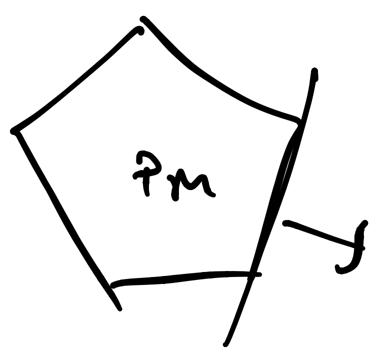
$\square$.

we'll skip facet proof; see pdf.

Facets of P_M

- which of the $2^{|E|}$ inequalities

  define facets of P_M?



- For simplicity,

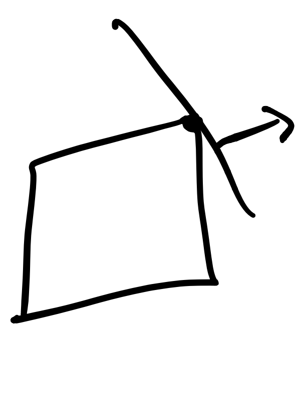
  ( ).

$\Rightarrow$

i.e.

- Rank constraints? $x(S) \le r(S)$.

  ▷ if S not closed,



- If S <u>seperable</u>, i.e.

$\Leftarrow$

- **Fact**: $S \rightsquigarrow$ facet $\Leftrightarrow$

<u>Proof omitted</u>

- **E.g.** <u>Graphic matroid</u> $M(G)$;

▷ **Exercise** $F \subseteq E$ inseparable $\iff$

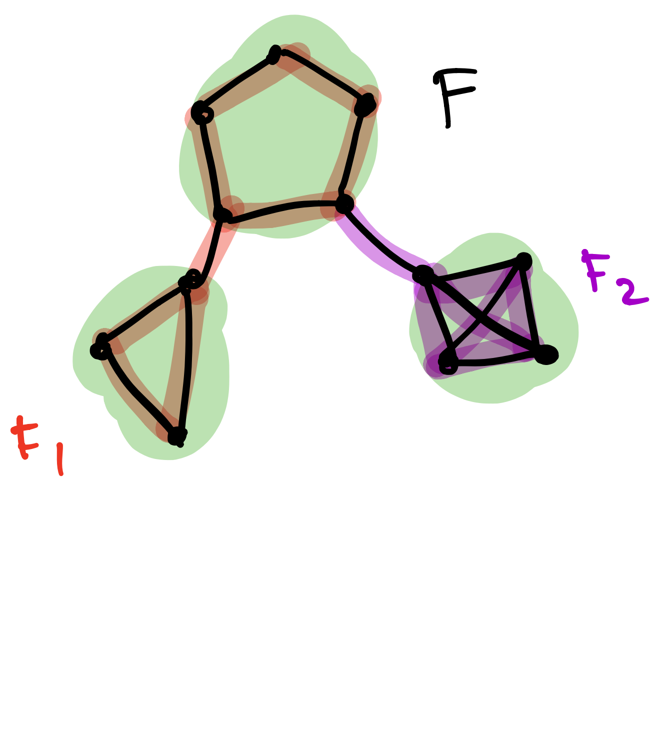
(V, F) is either

▷

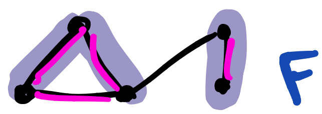
▷

E.g.

$|V| = 12$



▷ Span(F) =



▷ Thus F closed & inseparable

$\Leftrightarrow$

$\Rightarrow$ "Forest polytope" is minimally described by

$P = \{ x \in \mathbb{R}^E :$

$\}$.

"spanning tree" polytope:

$$P = \left\{ x \in \mathbb{R}^E : x(E) = |V| - 1 \right.,$$

$$\left. \right\}.$$